GRASP Laboratory

General Robotics & Autonomous Systems and Processes

Mechatronic Modeling and Design with Applications in Robotics

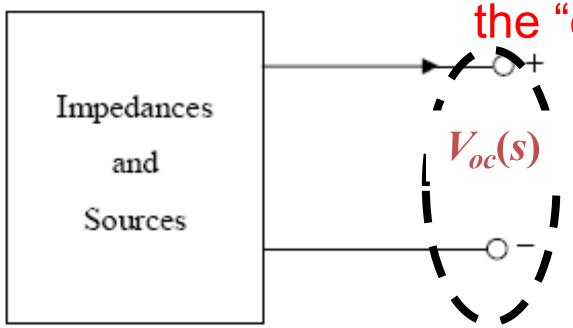
Transfer-Function Linear Graph

Outline

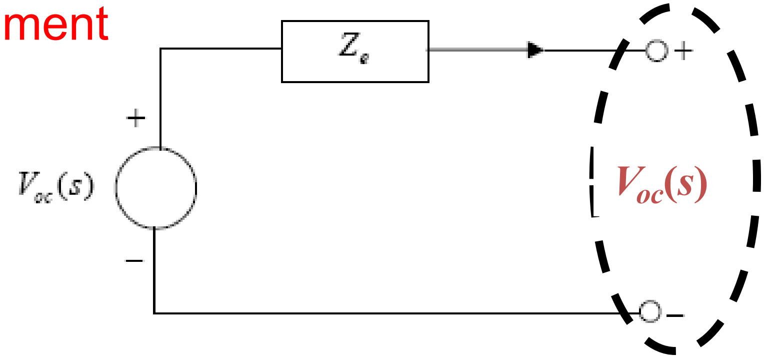
- Equivalent Circuits (Thevenin, Norton)
- Transfer Function Linear Graphs
- Linear Graph Reduction

Thevenin's Theorem for Electrical Circuits

Initial cut. Typically, the "output" element

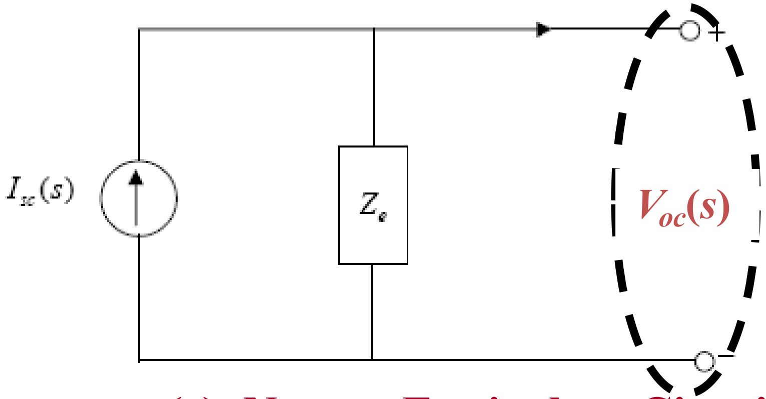


(a) Complex Circuit segment with impedances and sources



(b) Thevenin Equivalent Circuit

When do you prefer Thevenin; Norton?



(c) Norton Equivalent Circuit

Z_e = equivalent impedance with sources killed (i.e., voltage sources shorted and current source opened) = Thevenin impedance

$V_{oc}(s)$ = open-circuit voltage at cut
$I_{sc}(s)$ = short-circuit current at cut

***Note*: $V_{oc}(s) = Z_e I_{sc}(s)$** **Prove**

***Note*: Variables can be in Laplace or frequency domains**

***Note*: For multiple sources, use superposition (linear system)**

General Procedures for Mechanical Circuit Analysis Using Transfer Function Linear Graphs TFLGs

1. For each branch mark the Mobility function (not mechanical impedance)
2. Carry out linear graph analysis and reduction as if you are dealing with an electrical circuit, by using the following analogies:

Mechanical Circuit	Electrical Circuit Analogy
Mobility Function	Electrical Impedance
Force	Current
Velocity	Voltage

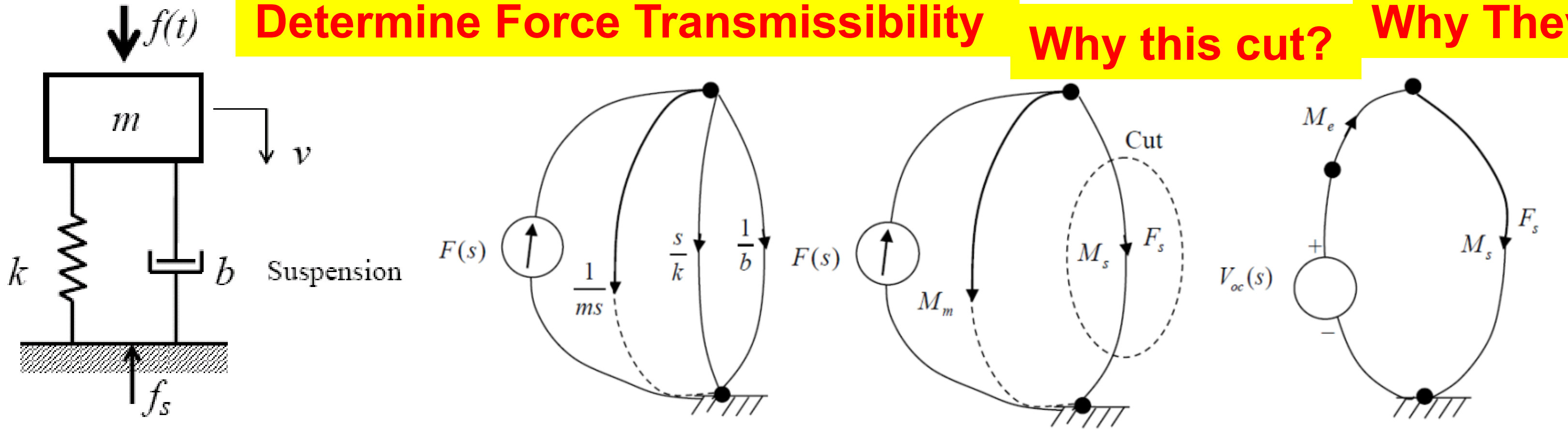
Also mark only the key variables on each branch.

In Particular: Think of the electrical analogy.

1. For parallel branches: mobilities are combined by inverse relation ($M = \frac{M_1 M_2}{M_1 + M_2}$);Velocity is common; Force is divided inversely to branch mobilities
2. For series branches: Mobilities add ($M = M_1 + M_2$); Force is common; velocity is divided in proportion to mobility
3. Killing a force source means open-circuiting it (so, transmitted force = 0)
4. Killing a velocity source means short-circuiting it (so, velocity across = 0)

Example 1: Ground-based Mechanical Oscillator (Revisited)

Determine Force Transmissibility | **Why this cut?** | **Why Thevenin?**



Physical System | **TF Linear Graph** | **Reduced TF LG** | **Thevenin Equiv. LG**

Note: $M_m = \dfrac{1}{ms}$; $M_s = \dfrac{\frac{s}{k} \times \frac{1}{b}}{\frac{s}{k} + \frac{1}{b}} = \dfrac{s}{bs + k}$

In TFLG, we show only the Mobility of each link (not the variable pair). Show only some key variables (e.g., input and output).

Cut suspension for force transmissibility ➔ Use Thevenin equivalence

Equivalent mobility (kill source ➔ "open" the link) = $M_e = M_m$

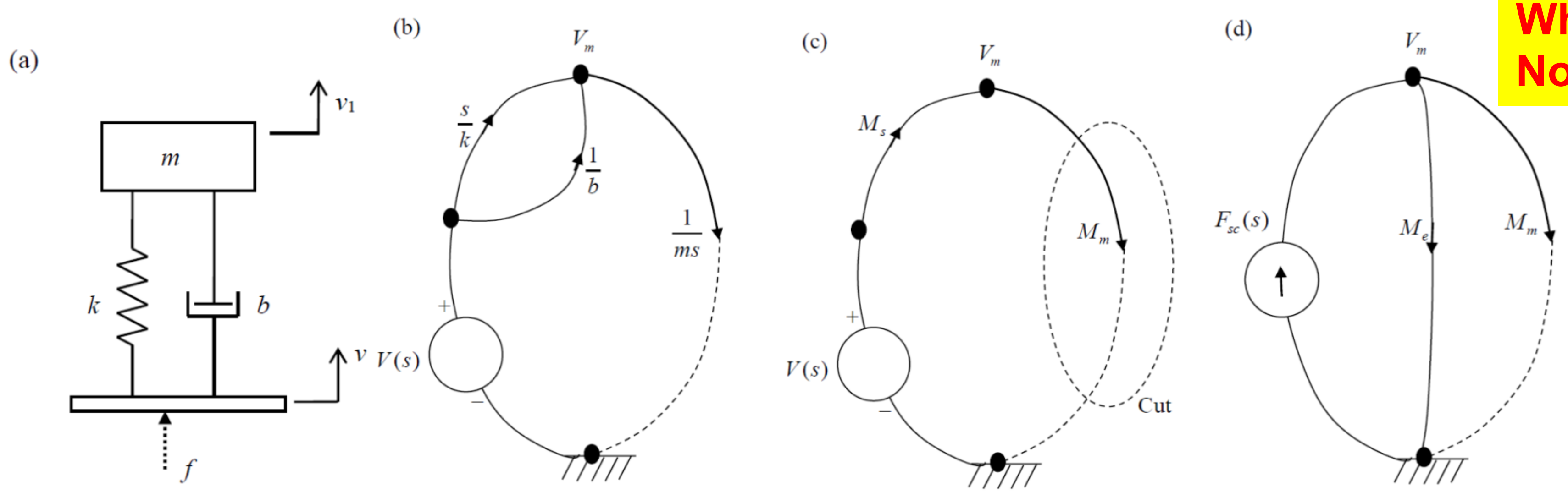
Open-circuit velocity $V_{oc}(s) = M_m F(s)$

From Thevenin LG, the transmitted force $F_s(s) = \dfrac{V_{oc}(s)}{M_e + M_s} = \dfrac{M_m F(s)}{M_m + M_s}$

➔ **Force transmissibility** $T_f = \dfrac{M_m}{M_m + M_s}$

Example 2: Oscillator with Support Motion (Revisited)

Why Norton?



Physical System | **Transf. Func. LG** | **Reduced TF LG** | **Norton Equiv. LG**

Determine Motion Transmissibility | **Why this cut?**

Cut mass element for motion transmissibility ➔ Use Norton equivalence

Equivalent mobility (kill source; i.e., short it) = $M_e = M_s$

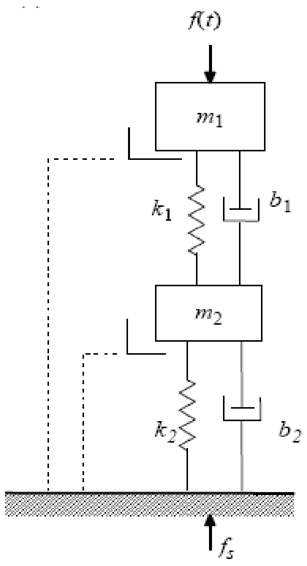
Closed-circuit force $F_{sc}(s) = \dfrac{V(s)}{M_s}$

From Norton LG, velocity at mass $V_m = \dfrac{M_e M_m}{M_e + M_m} F_{sc}(s) = \dfrac{M_s M_m}{M_s + M_m} \dfrac{V(s)}{M_s} = \dfrac{M_m V(s)}{M_m + M_s}$

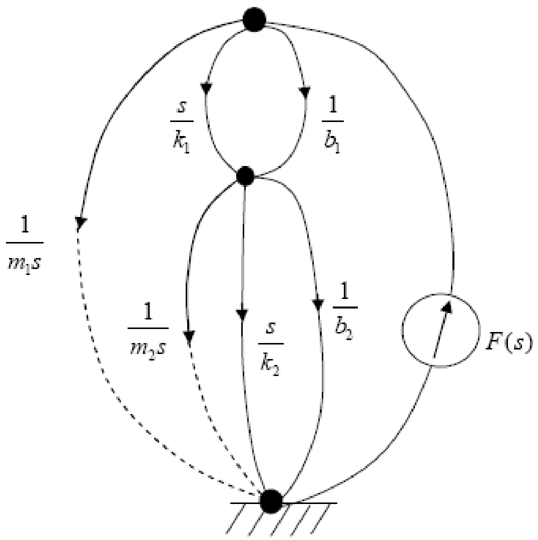
→ Motion transmissibility $T_m = \dfrac{M_m}{M_m + M_s}$ ← **Same as** T_f

Determine Force Transmissibility

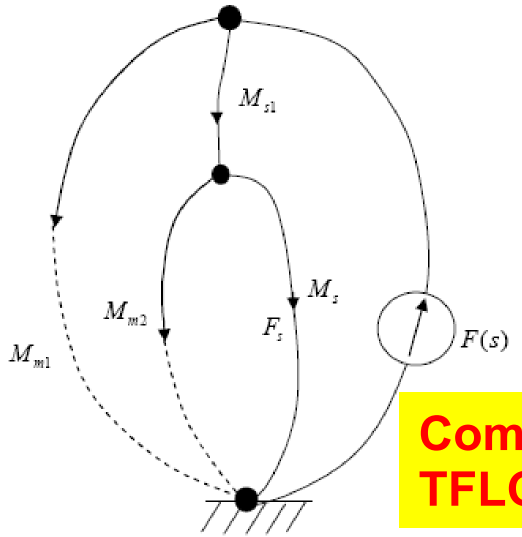
Example 3: Ground-based 2DOF Mechanical System (Revisited)



Mechanical System



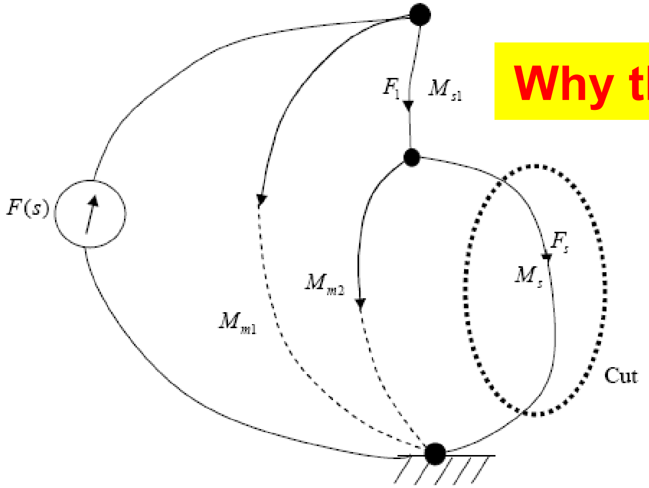
Linear Graph



Combined TFLG

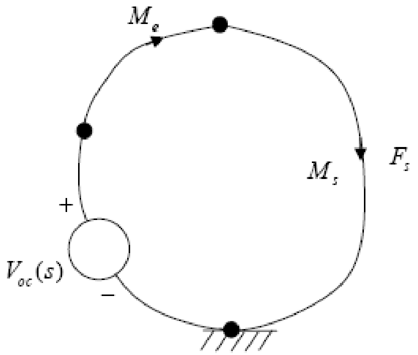
Linear graph showing suspension components

Why Thevenin?



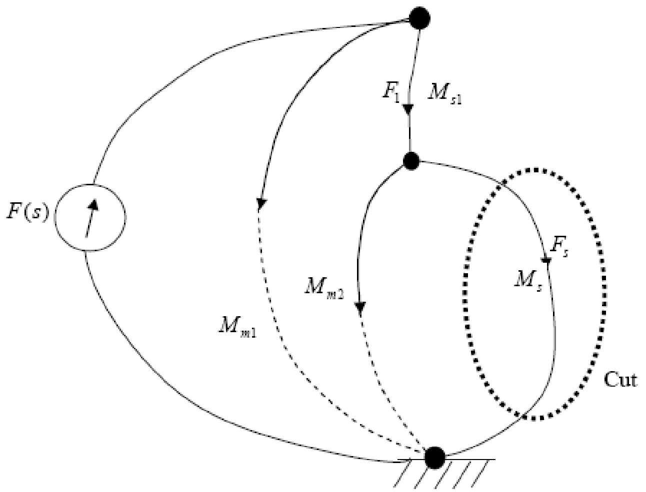
Why this cut?

Cutting Out the Suspension Unit of Interest

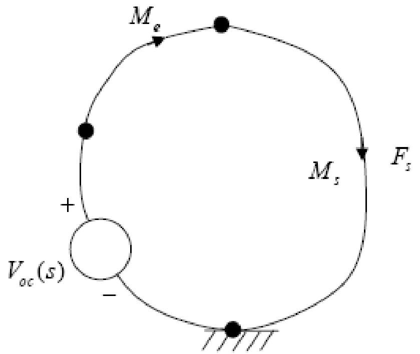


Thevenin Equivalent Circuit

Example 3 (Cont'd)



Cutting Out the Suspension Unit of Interest



Thevenin Equivalent Circuit

$$F_1 = \frac{M_{m1}}{(M_{s1}+M_{m2}+M_{m1})}F(s)\,;\quad V_{oc}(s) = M_{m2}F_1 = \frac{M_{m1}M_{m2}}{(M_{s1}+M_{m2}+M_{m1})}F(s)\,;\quad M_e = \frac{M_{m2}(M_{s1}+M_{m1})}{M_{m2}+(M_{s1}+M_{m1})}$$

$$F_s = \frac{V_{oc}(s)}{(M_e+M_s)} = \frac{M_{m1}M_{m2}}{(M_{s1}+M_{m2}+M_{m1})}F(s)\frac{1}{\left[\frac{M_{m2}(M_{s1}+M_{m1})}{M_{m2}+(M_{s1}+M_{m1})}+M_s\right]}$$

$$F_s = \frac{M_{m1}M_{m2}F(s)}{M_{m2}(M_{s1}+M_{m1})+M_s(M_{s1}+M_{m2}+M_{m1})} \rightarrow T_f = \frac{M_{m1}M_{m2}}{M_{m2}(M_{s1}+M_{m1})+M_s(M_{s1}+M_{m2}+M_{m1})}$$

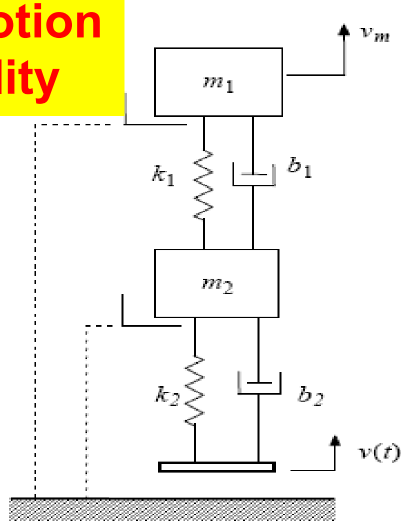
Show that this is identical the previous result.

***Note*: In the first step, force is divided inversely of Mobility. Why?**

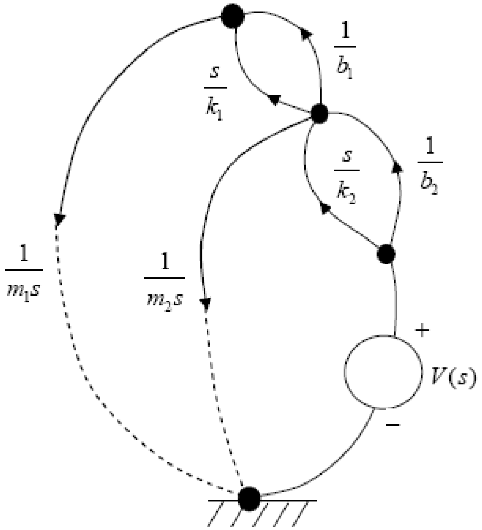
Or, can use, force is divided in proportion to Impedance.

Example 4: 2DOF Mechanical System with Support Motion (Revisited)

Determine Motion Transmissibility

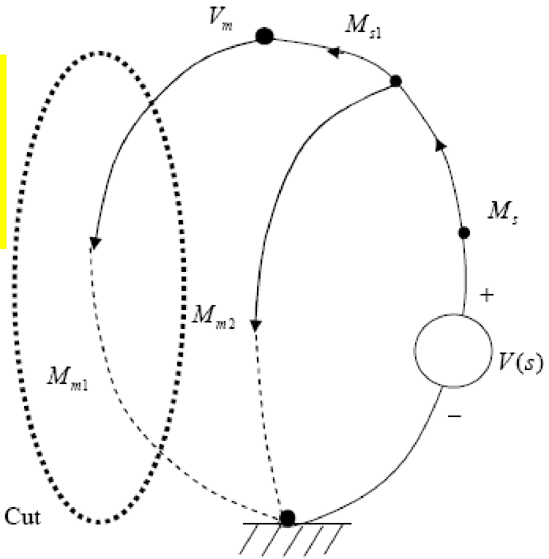


Mechanical System



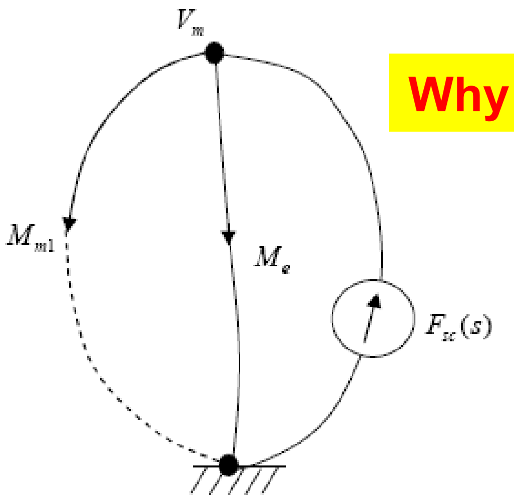
Linear Graph

Why this cut?



Linear Graph Showing Suspension Components

Combined TFLG

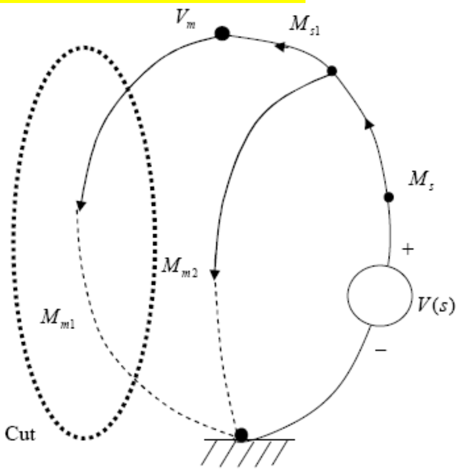


Norton Equivalent Circuit

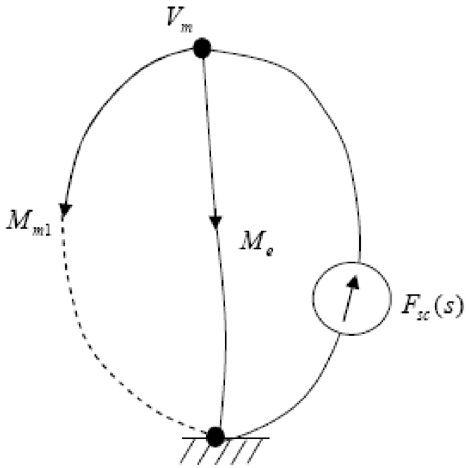
Why Norton?

Example 4 (Cont'd))

Combined TFLG



Transfer Function Linear Graph



Norton Equivalent Circuit

After shorting the cut, force provided by velocity source = $\dfrac{V(s)}{[\dfrac{M_{s1}M_{m2}}{(M_{s1}+M_{m2})}+M_s]}$

➔ **Short-circuit force** $F_{sc}(s)=\dfrac{V(s)}{[\dfrac{M_{s1}M_{m2}}{(M_{s1}+M_{m2})}+M_s]}\dfrac{M_{m2}}{(M_{s1}+M_{m2})}=\dfrac{M_{m2}}{M_{s1}M_{m2}+M_s(M_{s1}+M_{m2})}V(s)$

After killing source, equivalent mobility $M_e=M_{s1}+\dfrac{M_{m2}M_s}{M_{m2}+M_s}=\dfrac{M_{s1}(M_{m2}+M_s)+M_{m2}M_s}{M_{m2}+M_s}=\dfrac{M_{s1}M_{m2}+M_s(M_{s1}+M_{m2})}{M_{m2}+M_s}$

From Norton circuit, mass velocity $V_m=\dfrac{M_{m1}M_e}{(M_{m1}+M_e)}F_{sc}(s)$; ***Note***: $F_{sc}(s)M_e=\dfrac{M_{m2}}{(M_{m2}+M_s)}V(s)$

➔

$$V_m=\frac{M_{m1}}{\left[M_{m1}+\dfrac{M_{s1}M_{m2}+M_s(M_{s1}+M_{m2})}{M_{m2}+M_s}\right]}\frac{M_{m2}}{(M_{m2}+M_s)}V(s)=\frac{M_{m1}M_{m2}}{M_{m1}(M_{m2}+M_s)+M_{s1}M_{m2}+M_s(M_{s1}+M_{m2})}V(s)$$

$$=\frac{M_{m1}M_{m2}}{M_{m2}(M_{m1}+M_{s1})+M_s(M_{m1}+M_{s1}+M_{m2})}V(s)\ \rightarrow\ T_m=\frac{M_{m1}M_{m2}}{M_{m2}(M_{m1}+M_{s1})+M_s(M_{m1}+M_{s1}+M_{m2})}\ \leftarrow \text{same as } T_f$$

***Note*: In the first step, force is divided inversely of Mobility.**

Summary of Thevenin Approach for Mechanical Circuits

General Steps

1. Draw the linear graph for the system and mark the mobility functions for all the branches (except the source elements) + key variables
2. Simplify the linear graph by combining branches as appropriate (series branches: add mobilities; parallel branches; inverse rule applies for mobilities) and mark the mobilities of the combined branches
3. Based on the problem objective (e.g., determine a particular force, velocity, transfer function) determine which part of the circuit (linear graph) should be cut (i.e., The variable or function of interest should be associated with this part of the circuit)
4. Based on the problem objective establish whether Thevenin equivalence or Norton equivalence is needed (specifically: Use Thevenin equivalence if a T-variable to be determined, because → 2 series elements with a common T-variable; Use Norton equivalence if an A-variable to be determined, because → 2 parallel elements with a common A- variable)
5. Determine the equivalent source and mobility of the equivalent circuit
6. Using the equivalent circuit determine variable or function of interest

Justification for the Use of Equivalent Circuit and TFLG Approach

- **Associated techniques are well-established and fully-developed in the electrical domain**
- **Straightforward steps are followed**
- **Unified treatment (for multi-physics—See later) and systematic approach**
- **Linear graph representation is used (and only very few variables are used)**
- **Uses algebra (transfer functions) rather than calculus (differential equations)**

Equivalent TFLG for Multi-domain Systems

Equivalent TFLG of a Multi-domain TFLG

Method:

- **Decide which domain is converted (based objective; Typically, input domain is converted to output domain)**
- **Determine the Thevenin LG of subsystem to be converted (this is connected to "input branch" of two-port element—transformer or gyrator)**
- **For Thevenin: Apply equations (constitutive, loop, node) of two-port element → equivalent *A*-source and generalized impedance in series for the input domain. This becomes the "output branch" of the two-port element**

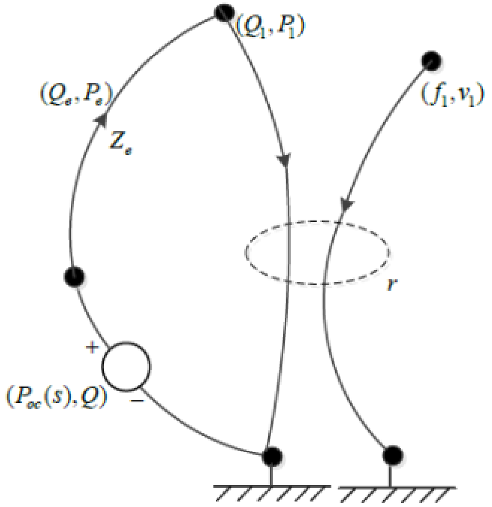
Equivalent TFLG of Multi-domain TFLG (Cont'd)

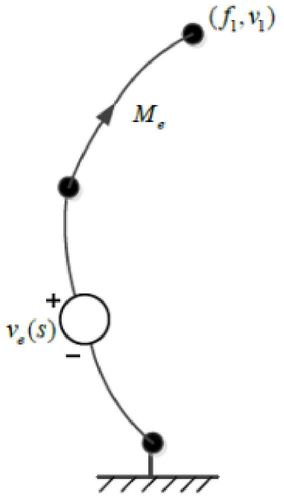
***Notes*:**

1. **In step 2, we may use Norton equivalent LG instead of Thevenin equivalent LG (Usually, Thevenin is easier)**
2. **In Step 3, we may determine the equivalent *T*-source and the equivalent generalized impedance in parallel (i.e., Norton. But Thevenin is easier)**

The final result will be the same with these choices. But the intermediate analysis will be different.

Transformer-coupled Systems





Transformer-coupled LG; Converted LG (Thevenin)

(Fluid to Mechanical Conversion)

Constitutive Equations :

$$v_1 = rP_1; \quad f_1 = -\frac{1}{r}Q_1; \quad P_e = Z_e Q_e$$

Loop Eq: $-P_1 - P_e + P_{oc}(s) = 0$

Node Eq: $Q_e - Q_1 = 0$

Other domain conversions are done in the same way.

Substitute:

$$v_1 = rP_1 = r[P_{oc}(s) - P_e] = rP_{oc}(s) - rZ_e Q_e = rP_{oc}(s) - rZ_e Q_1$$

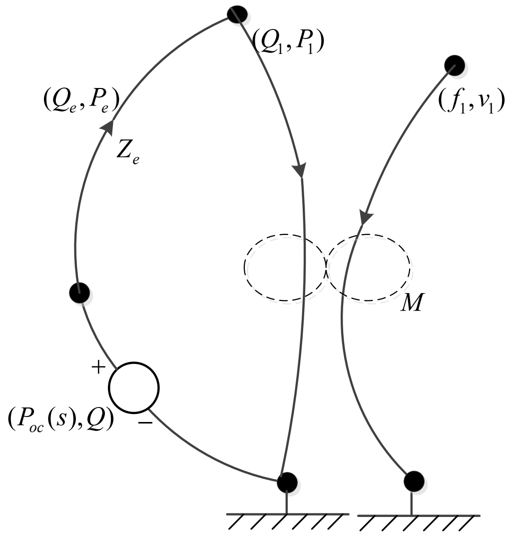
→ $v_1 = rP_{oc}(s) + r^2 Z_e f_1$

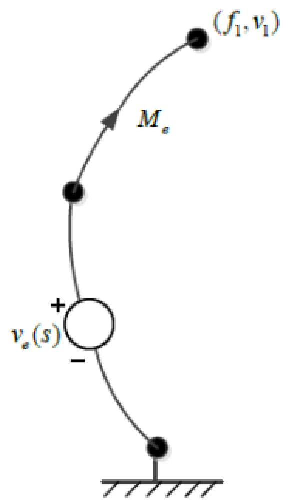
→ $v_1 = V_e(s) - M_e f_1$ **(because f_1 is reversed in the converted branch)**

Equivalent *A*-source: $V_e(s) = rP_{oc}(s)$

Equivalent generalized impedance: $M_e = r^2 Z_e$

Gyrator-coupled Systems





Gyrator-coupled LG; **Converted LG (Thevenin)**

(For Fluid to Mechanical Conversion)

Constitutive Equations:

$$v_1 = MQ_1;\ f_1 = -\frac{1}{M}P_1;\ P_e = Z_e Q_e$$

Loop Eq: $-P_1 - P_e + P_{oc}(s) = 0$

Node Eq: $Q_e - Q_1 = 0$

Other domain conversions are done in the same way.

Substitute:

$$v_1 = MQ_1 = MQ_e = \frac{MP_e}{Z_e} = \frac{M}{Z_e}[P_{oc}(s) - P_1] = \frac{M}{Z_e}[P_{oc}(s) + Mf_1]$$

$$\rightarrow v_1 = \frac{M}{Z_e}P_{oc}(s) + \frac{M^2}{Z_e}f_1 = V_e(s) - M_e f_1$$

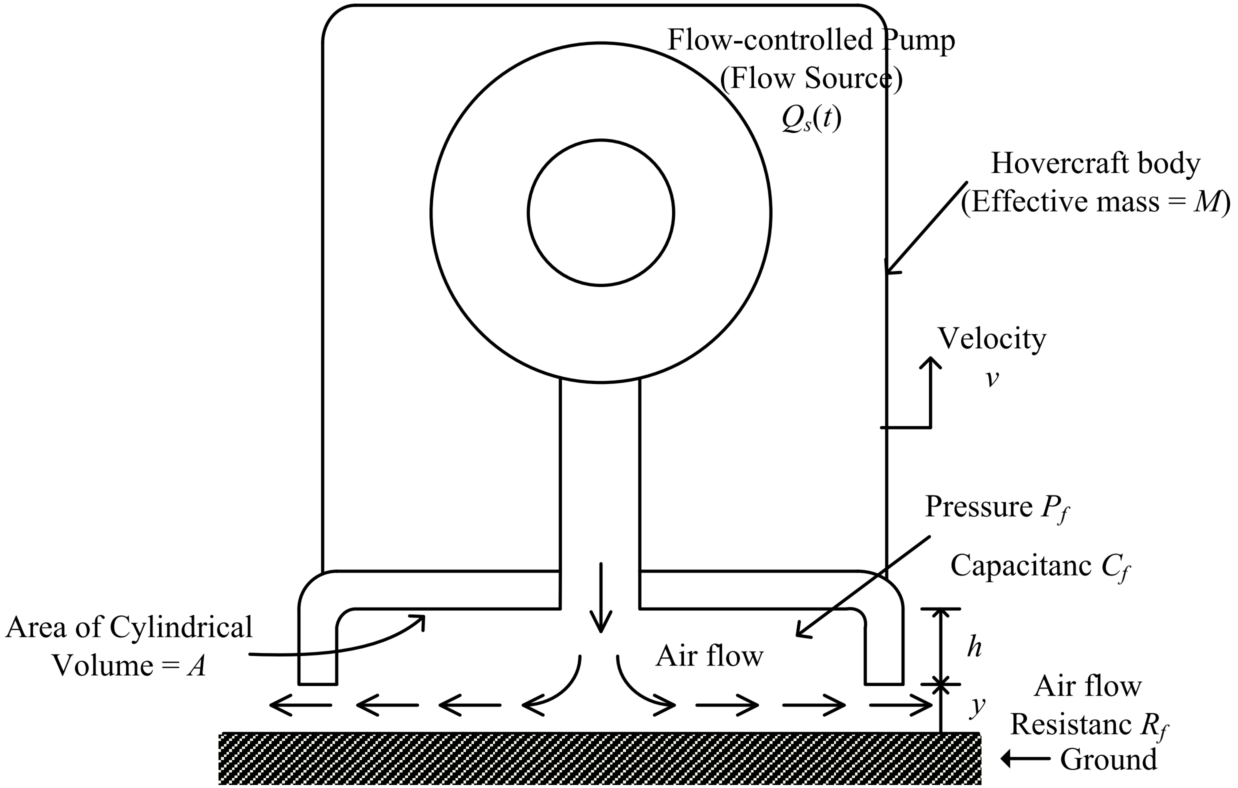
($-M_e$ because f_1 is reversed in the converted branch)

Equivalent *A*-source: $V_e(s) = \frac{M}{Z_e}P_{oc}(s)$

Equivalent generalized impedance: $M_e = \frac{M^2}{Z_e}$

Mixed-domain Example (Mechanical-Fluid)

(Heave motion (up and-down) of a Hovercraft)



Show that the transfer function is:

$$\frac{v}{Q_s} = \frac{R_f A}{R_f C_f M s^2 + M s + R_f A^2}$$

The End!!